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Efficient broadcasting with linearly bounded faults ☆

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Abstract

We consider the problem of broadcasting in the presence of linearly bounded number of transient faults. For some fixed $0<\alpha<1$ we assume that at most αi faulty transmissions can occur during the first i time units of the broadcasting algorithm execution, for every natural i. In a unit of time every node can communicate with at most one neighbor. We prove that some bounded degree networks are robust with respect to linearly bounded transmission faults for any $0<\alpha<1$, and give examples of such networks with logarithmic broadcast time, thus solving the open problem from Gąsieniec and Pelc (1998). We also show that our broadcasting algorithm is asymptotically optimal in terms of message complexity. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Broadcasting is one of the most fundamental tasks in network communication. Its aim is to transmit the information originally held in one node of the network (called the *source*) to all other nodes.

Recently, fault-tolerant broadcasting has become an extensively studied domain (for a survey see [4]). The goal is to effectively accomplish the communication task in spite of failure of some network components. The wide range of features necessary to fully describe the model (such as the communication mode, the type, number and duration of faults, etc.) offers a vast area for research.

Let us begin with the detailed description of the model considered in this paper. The network is synchronous – we assume the existence of a global clock. In a single time unit a node can communicate with at most one of its neighbors. During a fault-free transmission information can pass in both directions; a faulty transmission has no effect. Our algorithms are non-adaptive, i.e. all transmissions are scheduled in advance

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and do not depend on execution history. We consider the *linearly bounded* fault model. Given a constant $0<\alpha<1$ we assume that at most αi arbitrarily placed transient faulty transmissions can occur during the first i time units of the communication process.

For a fixed parameter $0<\alpha<1$, a given network $\mathcal{N}$, and a source node s, a broadcasting algorithm is called α-safe if it broadcasts information from the source s to all nodes of $\mathcal{N}$, whenever the number of faulty transmissions during the algorithm execution satisfies the above assumption with parameter α. A network in which α-safe broadcasting can be accomplished in time linear in its fault-free broadcasting time, for any source s and constant $0<\alpha<1$, is called *robust* with respect to linearly bounded transmission faults.

The linearly bounded fault model described above and previously used in [1, 5] in the context of searching with errors, has been studied by Gąsieniec and Pelc [3]. One of their results was the proof that the hypercube is robust with respect to linearly bounded transmission faults. They stated the following question as open: is there a bounded degree network for which α-safe broadcasting time is logarithmic for any constant $0<\alpha<1$? Our main result is the affirmative answer to this question.

(It might seem more natural to let the number of faults be proportional to the number of messages sent rather than to the number of time units. However, the results from [3] concerning the star graph imply that in this case a non-adaptive α-safe broadcasting algorithm requires an exponential number of messages, even in a complete network, for any $0<\alpha<1$. Thus, this variation of the fault model is not interesting.)

The rest of this paper is organized as follows. In Section 2 we present a general broadcasting algorithm and prove that it works fast in the linearly bounded fault model under some assumptions about the network connectivity. In Section 3 we apply this general method to show that constant degree multidimensional tori and wrap-around butterflies are robust with respect to linearly bounded transmission faults. In Section 4 we show that our algorithm for butterflies as well as the algorithms for complete graphs and hypercubes from [3] are, in fact, asymptotically optimal in terms of message complexity. Section 5 contains conclusions.

2. Broadcasting through perfect matchings

Let $G=(V,E)$ be the underlying graph of our network and assume that the set of its edges E can be partitioned into pairwise disjoint perfect matchings $M_1,\ldots,M_d$. Consider the following algorithm.

Algorithm. BROADCAST$(t,M_1,\ldots,M_d)$
repeat t **times**
    for $i = 1..d$ **do**
        for $(v,w)\in M_i$ **in parallel do**
            nodes v and w exchange information

Its execution time is clearly td.

Lemma 1. *Let $s, v_1, \ldots, v_d \in V$ (the nodes $v_1, \ldots, v_d$ are not necessarily different) and assume that there exist edge-disjoint paths between s and v_i for $i = 1, \ldots, d$, each path of length at most D. If the node s is initially informed and there are at most k transmission faults during the execution of the algorithm BROADCAST then after $dD + k$ time units at least one of the nodes $v_1, \ldots, v_d$ becomes informed.*

Proof. Each transmission fault on a single-path delays the information flow on this path for at most d time units. Since there are at most k faults on d paths, there is a path with at most $\lfloor k/d \rfloor$ faults. The information reaches its end in at most $d(D + \lfloor k/d \rfloor) \leqslant dD + k$ time units. □

The above lemma implies an upper bound on the broadcasting time with the linearly bounded transmission faults if each node of the network can be connected with the source by many short edge-disjoint paths.

Corollary 2. *Suppose that every node of the network is connected with the source by d edge-disjoint paths, each of length at most D. Then the algorithm BROADCAST is α-safe for $t \geqslant D/(1-\alpha)$.*

Proof. Denote $T = \lceil dD/(1-\alpha) \rceil$ and fix $v \in V$. It is enough to show that v becomes informed in at most T time units. The number of faults that may occur during this time does not exceed αT, so by Lemma 1 the information reaches v after $dD + \alpha T \leqslant T$ time units. □

3. The robust bounded degree networks

We will now apply the results of the previous section to prove that some popular constant degree networks are robust with respect to linearly bounded faults. First, we consider the multidimensional tori, then the wrap-around butterfly – the network of constant degree and logarithmic broadcast time.

3.1. The multidimensional tori

A k-dimensional torus $\mathscr{T} = \mathscr{T}(n_1, \ldots, n_k)$ for integers $n_i \geqslant 1$ is defined as follows. The nodes of the network are the points of the k-dimensional grid $\{0..n_1\} \times \cdots \times \{0..n_k\}$. Two points $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are connected by an edge if and only if x and y differ in exactly one coordinate, say i, and $x_i = y_i \pm 1 (\operatorname{mod} n_i + 1)$. Hence, each node is connected either to exactly one other node by an edge in dimension i – if $n_i = 1$, or to exactly two vertices, otherwise. The degree of every node of $\mathscr{T}$ equals $\Delta = 2k - |\{i\colon n_i = 1\}|$, and the diameter of $\mathscr{T}$ is $\Theta(\sum_{i=1}^{k} n_i)$. The proof of the following lemma can be found in [2]:

Lemma 3. *For each pair of nodes u and v of* $\mathscr{T}(n_1,\dots,n_k)$, *there are* Δ *edge-disjoint paths connecting u with v, each of length at most* $1+\sum n_i$.

If all integers $n_1,\dots,n_k$ are odd then there is an obvious partition of edges of $\mathscr{T}(n_1,\dots,n_k)$ into Δ perfect matchings. Thus, the Corollary 2 from Section 2 gives us the following theorem, which can be viewed as a generalization of the result from [3] concerning the ring network.

Theorem 4. *For any fixed k and odd integers* $n_1,\dots,n_k$, *a k-dimensional torus* $\mathscr{T}(n_1,\dots,n_k)$ *is robust with respect to linearly bounded faults.*

3.2. The butterfly

An r-dimensional *butterfly* has $(r+1)2^r$ nodes arranged in $r+1$ levels. Each node has a distinct label (w,i), where i is the *level* of the node $(0\leqslant i\leqslant r)$ and w is an r-bit number $b_0..b_{r-1}$ denoting the *row* of the node (in this subsection we use the convention of writing binary numbers with the least significant bit at the beginning). Two nodes (w,i) and (w',i') are connected by an edge if and only if $i'=i+1$ and either $w=w'$ (*straight* edge) or w and w' differ precisely in bit i (*cross* edge). In an r-dimensional *wrap-around butterfly* $\mathscr{B}_r$ the level 0 and r nodes in each row are assumed to be the same node. The degree of each node of $\mathscr{B}_r$ equals 4 for any $r>1$ and the diameter is $\Theta(r)$. If r is even then the four sets

$$M_1=\{\text{straight edges connecting levels } 2k \text{ with } 2k+1, \text{ for } k=0,\dots,r/2\},$$

$$M_2=\{\text{cross edges connecting levels } 2k \text{ with } 2k+1, \text{ for } k=0,\dots,r/2\},$$

$$M_3=\{\text{straight edges connecting levels } 2k-1 \text{ with } 2k, \text{ for } k=1,\dots,r/2\},$$

$$M_4=\{\text{cross edges connecting levels } 2k-1 \text{ with } 2k, \text{ for } k=1,\dots,r/2\}$$

partition all the edges of $\mathscr{B}_r$ into four perfect matchings.

In order to simplify our discussion of the structure of $\mathscr{B}_r$ we show that it is *symmetric*, i.e. for each pair of its nodes x and y there exists an automorphism of $\mathscr{B}_r$ that maps x on y.

Lemma 5. $\mathscr{B}_r$ *is symmetric.*

Proof. It is enough to show an automorphism that maps the node $(0,0)$ onto $(w,0)$, for any $w<2^r$, and another one that maps $(0,0)$ onto $(0,1)$. It is not hard to check that the following equations:

$$R_w((v,i))=(v\,\mathbf{xor}\,w,i),$$

$$L((v,i))=(v\,\mathbf{shr}\,1,i+1)$$

for $0 \leqslant v < 2^r, 0 \leqslant i < r$, define, in fact, the required automorphisms. (The operator **xor** denotes the bitwise exclusive-or; **shr** is the cyclic shift.) □

Let us introduce some notational conventions used throughout the remaining part of this subsection. The straight and cross edges are labeled 0 and 1, respectively. Traversing an edge with label b "from left to right", i.e. moving from level i to $i+1$, will be denoted by $\overrightarrow{b}$; traversing it in the opposite direction will be denoted by $\overleftarrow{b}$. An arrow over a string of 0s and 1s means putting the same arrow over each element of this string. v^{R} is the reverse of a string v, while $neg(v)$ is the complement of v. The abbreviation for k repetitions of string v is v^k. Finally, $tr(v)$ denotes a string v with all the rightmost (unsignificant) zeros deleted; e.g. $tr(0100) = 01$.

Although we will not prove the exact analog of Lemma 3 for the wrap-around butterfly, the following fact is close enough.

Lemma 6. *Consider the graph $\mathscr{B}_r$ for $r > 1$.*
(a) *There are four edge-disjoint paths of length $O(r)$, joining the node $(w, 0)$ with the node (w, i), for any $0 < i < r$.*
(b) *There are four edge-disjoint paths of length $O(r)$, joining the node $(0, 0)$ with nodes from the row w, for any $0 < w < 2^r$.*

Proof. The lemma can be checked directly for $r = 2$, so we assume $r > 2$.
(a) Take the paths $(\overrightarrow{0})^i, (\overleftarrow{0})^{r-i}, (\overrightarrow{1}\,\overleftarrow{0}\,\overrightarrow{1})^i$, and $(\overleftarrow{1}\,\overrightarrow{0}\,\overleftarrow{1})^{r-i}$, starting from $(w, 0)$.
(b) Let us first consider the case $w = 2^r - 1$. The paths we need are $(\overrightarrow{1})^r, (\overleftarrow{1})^r, \overrightarrow{0}\,(\overrightarrow{1})^r\,\overleftarrow{0}$, and $\overleftarrow{0}\,(\overrightarrow{1})^{r-1}\,\overrightarrow{0}$.

Now assume $0 < w < 2^{r-1}$. Two natural paths are $p_1 = tr(\overrightarrow{w})$ and $p_2 = tr(\overleftarrow{w^{\mathrm{R}}})$. The other two paths are constructed by descending to the lowest row $2^r - 1$ and then climbing up to row w. More precisely, the paths p_1 and p_2 leave exactly two edges incident with the node $(0, 0)$ unused. We descend to the last row by the previously described paths that begin with these edges, say p_{3a} that begins with $\overrightarrow{b}$ and p_{4a} that begins with $\overleftarrow{b'}$, for some $b, b' \in \{0, 1\}$. The paths are disjoint with p_1 and p_2 unless $w = 2^{r-1}$ (we will handle this case separately), and their common endpoint is $(2^r - 1, 0)$. Then we extend either the path p_{3a} with $tr(\overleftarrow{neg(v^{\mathrm{R}})})$ and p_{4a} with $tr(\overrightarrow{neg(v)})$ (if $b = b' = 1$), or the path p_{3a} with $tr(\overrightarrow{neg(v)})$ and p_{4a} with $tr(\overleftarrow{neg(v^{\mathrm{R}})})$ (otherwise). These paths are edge-disjoint with each other, but they may contain a segment traversed twice in both direction, which should be deleted. Fig. 1 illustrates this construction.

It remains to show two additional paths connecting the node $(0, 0)$ with the row 2^{r-1}. These are $(\overrightarrow{1})^{2r-1}$ and $\overleftarrow{0}\,(\overleftarrow{1})^{2r-2}$. □

Now we are ready to prove the following theorem.

Theorem 7. *The wrap-around butterfly $\mathscr{B}_r$ is robust with respect to linearly bounded faults, for any even r.*

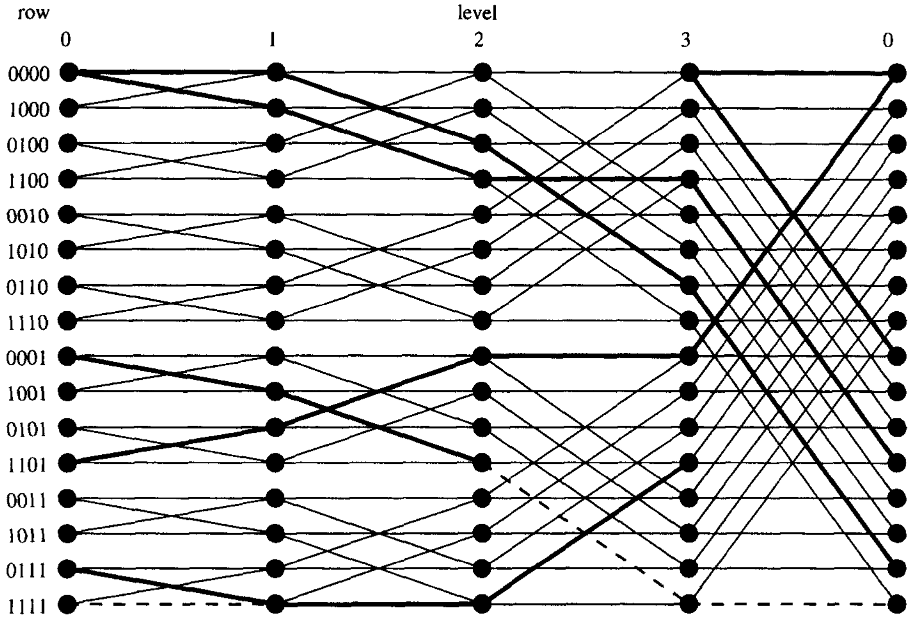

Fig. 1. Four edge-disjoint paths that connect the node (0,0) with row 1101 (dashed lines denote segments traversed twice and thus excluded from the path).

Proof. We will show that the algorithm BROADCAST applied to $\mathscr{B}_r$ is α-safe for $t \geqslant D(\lceil(1-\alpha)^{-2}\rceil + \lceil(1-\alpha)^{-1}\rceil)$, where $D = \mathrm{O}(r)$ is an upper bound on the length of the paths from Lemma 6 (actually, $D \leqslant 3r$).

Because of symmetry of $\mathscr{B}_r$ (Lemma 5) w.l.o.g. we can assume that $(0,0)$ is the source node. Fix a node (w,i) and take $T' = 4D(\lceil(1-\alpha)^{-2}\rceil + \lceil(1-\alpha)^{-1}\rceil)$. It is enough to prove that after T' time units node (w,i) becomes informed. By Lemma 6(b) there are four edge-disjoint paths connecting the source with nodes of row w. Following the proof of Corollary 2 we conclude that at least one of these nodes becomes informed after $T = 4D\lceil 1/(1-\alpha)\rceil$ time units. Again, by Lemma 5 we can assume that the informed node is $(w,0)$. By Lemma 6(a) this node is connected with (w,i) by four edge-disjoint paths, each of length at most D. There are at most $\alpha T' \leqslant T' - (T + 4D)$ faults during the first T' time steps of the algorithm, so by Lemma 1 node (w,i) becomes informed after at most $T + 4D + \alpha T' \leqslant T'$ time steps. $\square$

4. Lower bound for message complexity

A natural lower bound for the number of messages sent during broadcasting in an n-node network is clearly $\Omega(n)$. It is easy to see that our broadcasting algorithm for a wrap-around butterfly as well as the algorithms for a complete graph and a hypercube from [3] use $\Theta(n \log n)$ messages. It turns out that this message complexity is, in fact, asymptotically optimal, as the following theorem shows.

Theorem 8. *Performing α-safe broadcasting in an n-node network requires at least $(\alpha^2/2)n\log n$ messages, for any fixed $0<\alpha<1$ and sufficiently large n.*

Proof. Let us fix some broadcasting scheme $\mathcal{S}$ which uses less than $(\alpha^2/2)n\log n$ and suppose that it works in time $T\geqslant\log n$. Let A be the set of nodes which get less than $\alpha^2\log n$ messages during the whole communication process. Clearly, $|A|\geqslant n/2$ (otherwise the total number of messages would exceed $(\alpha^2/2)n\log n$), so $|A|\geqslant 2^{\alpha\log n}$ for sufficiently large n.

We shall now exhibit a linearly bounded fault pattern which assures that at least one node does not receive information and so the scheme $\mathcal{S}$ is not α-safe. The first $\alpha\log n$ rounds are fault-free. The total number of nodes which have been informed so far, is at most $2^{\alpha\log n}$, so there still remains at least one uninformed node p in A. Blocking all further transmissions concerning p assures that the information will never reach p. Since there are less than $\alpha^2\log n$ such transmissions, the total number of faults will not exceed the allowed fraction. □

5. Conclusions

We studied the time of broadcasting in some constant degree networks, under the assumption that at most αi transmission failures occur in the first i steps of broadcasting, for any natural i and a fixed $0<\alpha<1$. We presented a simple generic algorithm which provides asymptotically time-optimal α-safe broadcasting in a wide class of bounded degree networks, including tori with even number of nodes in each dimension. Our main result was the proof that $2r$-dimensional wrap-around butterfly admits logarithmic α-safe broadcasting time for any $0<\alpha<1$ and $r>0$, which gives a positive answer to the open problem from [3]. The same result can also be obtained for $2r$-dimensional CCC, yielding an example of degree 3 network which is robust with respect to linearly bounded faults. Finally, we proved that the presented algorithm as well as the algorithms from [3] meet the lower bound for message complexity.

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